

Chance Constraints Within Linear Programming

by

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# **Abstract**

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Within linear programming, chance constraints offer a method for solving problems that contain random variables within the objective or constraints. This thesis will provide an overview of linear programming and chance constraints with random variables that can be modeled with the normal distribution. A discussion of both single and joint constraints, the history of the development of the method, and survey of current applications is included.

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# Chapter 1: Introduction to Linear Programming

To begin our discussion of chance constraints, we must first start with a firm understanding of linear programming, as well as the more general term of operations research. Overall, operations research refers to using a procedural method to discover the most efficient way for an operation to be performed. Similar to the scientific method, steps are performed to conclude mathematically and definitively that the final method is in fact the best course of action for the operation. Types of operations that this may be referring to include business operations or wartime supplies. The cementation of operations research occurred during World War II, when the need to allocate war supplies in the most efficient method possible was an urgent need, and new processes were created to find these optimal solutions. Following the end of the war, operations research was then brought to other fields outside of the military, and its broad range of potential uses has lead to its use in almost all fields [6].

Many of the problems that are solved by operations research fall into the more specific category of linear programming. Linear programming is a method to solve problems of operations research that contain multiple constraints and an objective, with all functions being of a linear nature.

As a simplistic example, these functions often include several variables representing quantities of products to produce, with limited resources (labor, raw materials, space, etc.) being represented by the constraints, with the value of each variable quantity shown in the objective function. This is known as a resource-allocation problem. Constraints for these problems can range from the constraints on each separate resource, constraints on the amount of the resources together, a particular proportionality of resources needing to be maintained, and

so on. Problems also often contain non-negativity constraints to ensure final variable quantities are positive. The goal of this particular form of linear programming problem is to find the combination of resources that fit within the constraints, and will maximize the objective function. Note however, that though objective functions often are maximized due to often being representative of profit, the problem may instead call for the function to be minimized, depending on the context of the problem. The outline of the equations for this problem is shown below for  $n$  products and  $m$  constraints. The below notations for both linear programming and chance constraints are adapted from those found in Hillier and Lieberman [6].

- The variables:  $x_j$  represents the quantity of the  $j^{th}$  product to produce,  $j = 1, \dots, n$
- The constraints:
  - Resource constraints, with  $a_{i,j}$  representing the amount of resource  $i$  needed for one unit of product  $j$ , and  $b_i$  representing the amount of resource  $i$  available:

$$\sum_{j=1}^n a_{i,j}x_j \leq b_i, \quad i = 1, \dots, m$$

- Non-negativity constraints, to ensure negative quantities are excluded:  $x_i \geq 0$ ,  $i = 1, \dots, n$
- The objective to be maximized, with  $c_j$  representing the value of one unit of the  $j^{th}$  product to produce:

$$Z = \sum_{j=1}^n c_jx_j$$

## A Simple Example

When linear programming problems contain only two variables, they can be viewed and solved graphically, which gives an excellent visual for the process behind solving linear programming problems. Here is the set up where we have used the more traditional  $x$  and  $y$  to

represent the variables.

The variables:

- $x$ : Quantity of product 1
- $y$ : Quantity of product 2

The Constraints:

- $a_{1,1} x + a_{1,2} y \leq b_1$
- $a_{2,1} x + a_{2,2} y \leq b_2$
- $x, y \geq 0$

Objective: Maximize  $Z = c_1x + c_2y$

When graphing two variable linear programming problems, the line created by the equation for each constraint function acts as a boundary that a solution must be within. Some constraints may already be satisfied by other constraints that are more strict. Together, these functions should create a complete boundary around an inner range, with this inner range representing the range of possible solutions. This area is called the feasible region, and all possible solutions within the region are considered feasible solutions that satisfy all constraints. Solutions not within this bounded region fail to satisfy at least one of the constraints, and these are considered infeasible solutions existing within the infeasible range.

To determine the optimal solution from within the feasible region, the objective function must then be added to the graph. For each inputted value of  $Z$ , we can see the combinations of  $x$  and  $y$  that lie along the slope of  $Z$  and result in this value, and which of these combinations exist in the feasible and infeasible regions. Our  $Z$  value can increase until it only contains one solution within the feasible region, or lies along an edge of the feasible region. This shows where the optimal solution to our problem lies, or if a range of solutions along an edge may all result in the optimal solution. This is known as the corner point feasible (CPF) solution. The optimal solution to a problem lies at a corner of the feasible region.



**Two Variable Example:** A book manufacturer produces both hard cover and paperback versions for a certain book. Both versions of the book take their printers 12 minutes to produce all pages to the book, and each of their five machines can only be turned on for 6 hours a day, for a total of 1,800 minutes of possible printing time per day. The producer also only has labor available to finalize the production of up to 100 hard cover copies each day, and up to 120 paper cover copies each day, each unaffected by one another. If paperback versions will be bought for \$3 each and hard cover version for \$5, what is the optimal number of paperback and hardcover versions of the book that the producer should make each day to maximize profit, and what is that profit?

**Solution:** Begin by setting up the example as we had shown previously.

The Variables:

- Number of paperback books to be made:  $x$
- Number of hard cover books to be made:  $y$

The Constraints:

- Amount of daily printing time:  $12x + 12y \leq 1,800$
- Labor available for book  $x$ :  $x \leq 120$
- Labor available for book  $y$ :  $y \leq 100$
- Non-negativity constraints:  $x, y \geq 0$

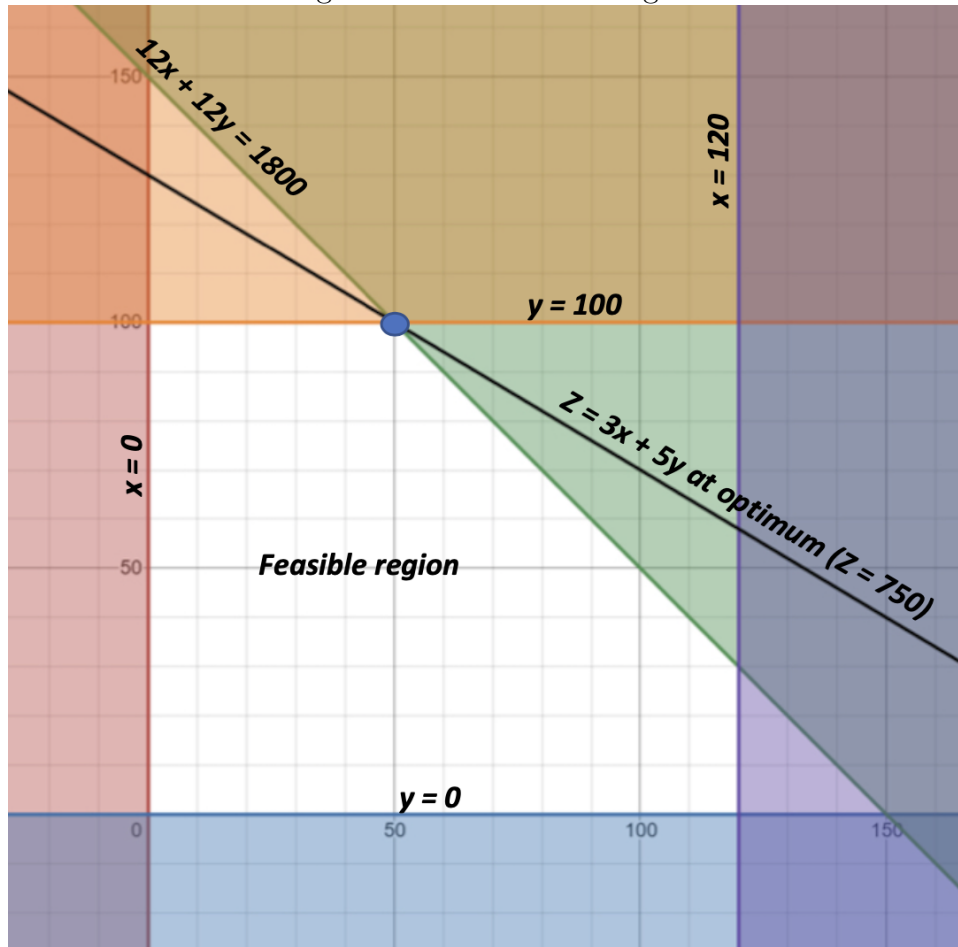
Objective: Maximize Revenue  $Z = 3x + 5y$

We will now be able to input each of these functions into a graphing calculator, preferably Desmos, to see where our optimal solution will lie. Begin by opening a blank graphing space within Desmos. It is often easier to visualize our feasible region when the surrounding region is colored in and the feasible region is shown as empty space, so it is recommended to write functions using the opposite sign. For example, the functions  $x \geq 0$  and  $y \geq 0$  allows for the areas that satisfy these constraints to be left blank, rather than writing it as  $x \leq 0$  and

$y \leq 0$ . Once this has been done for each of the constraint functions, we now see the feasible region fully surrounded by boundaries on each side. Now, the objective function will be added. When adding the third variable of  $Z$  to the function, verify that  $Z$  will be added as a slider. This will allow you to manually adjust the value of  $Z$  to see what the maximum value that  $Z$  can be, while still lying within the feasible region. Note that it will often be required to adjust what the maximum value we can input into  $Z$ , as the values in this equation are moderately large. We can go ahead and adjust this value to 750.

As we begin to move our slider throughout the feasible region, we eventually see that the last point that the objective function is within the feasible region before moving out of it is at the corner point between our functions  $y \leq 100$  and  $12x + 12y \leq 1800$ . This represents what the maximum value that our objective function can produce while staying within our given constraints, giving us our optimal solution. We see when clicking on this point that our optimal solution is to produce 50 paper back books and 100 hard cover books each day, resulting in a daily profit of our  $Z$  value, \$750. We can see the final graph of our feasible region with its optimal solution below in Figure 1.

Figure 1: The Feasible Region



## Solving Higher Dimension Problems

As problems become more complicated (or contain any number of variables greater than 2), a graph is no longer a feasible way to determine the solution to a linear programming problem. A common method for solving more complex problems relatively efficiently is the Simplex Method. Though this method can be performed by hand, it is typical, especially after reaching a certain level of complexity, to instead solve using a program such as Excel, which is what we will demonstrate below. Using Excel's Solver feature, determining and inputting the objective functions, constraints, and variables into Excel will allow for Excel to compute all calculations and find the optimal solution to a linear programming problem without any additional computing from the person solving. Below is an example of a linear

programming problem being solved using Excel 365 for Windows 10.

**Four Variable Example:** A car shop sells four different types of wheels, with each increasing in price as quality and size increases. The smallest size (Tire A) sells for \$60, with the others being \$90, \$150, and the most expensive being \$240 (referred to as Tires B, C, and D, respectively). The shop can only buy and store up to 100 total tires each week, but the store wants to hold at least 5 of each tire type at the start of each week. Certain producers can only make certain tire types, with the producer of Tires A and B having 70 labor hours in total available each week for their production, while Tires C and D have 150 hours available each week from their producer. Each Tire A requires one hour of labor, Tire B requires two hours, Tire C requires 3 hours, and Tire D requires 4 hours. Find the number of each tire type that should be stored at the tire shop each week that satisfies all constraints, while maximizing profit.

**Solution:** First, we must set up our problem. Below are our list of variables, our objective function, and each of our constraints.

Variables: Number of each product to make

- Tire A:  $x_1$
- Tire B:  $x_2$
- Tire C:  $x_3$
- Tire D:  $x_4$

Objective (Maximize profit):  $Z = 60x_1 + 90x_2 + 150x_3 + 240x_4$

Constraints:

- At least 5 of A:  $x_1 \geq 5$
- At least 5 of B:  $x_2 \geq 5$
- At least 5 of C:  $x_3 \geq 5$
- At least 5 of D:  $x_4 \geq 5$

- At most 100 tires can be stored:  $x_1 + x_2 + x_3 + x_4 \leq 100$
- Labor hours for A and B:  $x_1 + 2x_2 \leq 70$
- Labor hours for C and D:  $3x_3 + 4x_4 \leq 150$

We will now insert these variables and equations into Excel in order to solve.

In a new Excel sheet, use cells A1:A4 to list names for each of the four variables (i.e. write “x1” in Cell A1, “x2” in Cell A2, etc.). The cells to their right, B1:B4, are what the program recognizes as representing the four variables, with their names listed to their left for our understanding. In a cell further below, say cell B8, write the objective equation

$$= 60 \cdot B_1 + 90 \cdot B_2 + 150 \cdot B_3 + 240 \cdot B_4.$$

This shows our objective function with our variables represented by the cells that Excel considers to be the variables, B1:B4. We will then list our six constraint functions below. In cells B10 through B16, we will list the left hand side of each of our six equations, with Tire A again being referenced as B1 and so on. We will then use cells C10 through C16 to list the inequalities of each of our six constraints. Finally, cells D10 through D16 will have the right hand side of each of our equations listed. See Figure 2.

Figure 2: The set up in Excel

	A	B	C	D	E
1	x1				
2	x2				
3	x3				
4	x4				
5					
6					
7					
8	Objective	=60*B1 + 90*B2 + 150*B3 + 240*B4			
9					
10	Constraints	=B1	>=	5	Minimum tire amount of Tire A
11		=B2	>=	5	Minimum tire amount of Tire B
12		=B3	>=	5	Minimum tire amount of Tire C
13		=B4	>=	5	Minimum tire amount of Tire D
14		=B1+B2+B3+B4	<=	100	Maximum total tire amount
15		=B1+2*B2	<=	70	Maximum labor amount for Tires A & B
16		=3*B3 + 4*B4	<=	150	Maximum labor amount for Tires C & D
17					
18					
19					

We will now input this information within the Solver function in order for Excel to solve the problem. After clicking on “Solver” within the Data window a panel appears (Figure 3). We will then select cell B8 within “Set Objective,” as that is our objective function. Next, because we are maximizing our objective function, we select “Max” after the prompt “To.” We then select cells B1:B4 as “By Changing Variables Cells” to show that these cells will represent the variables that will be changed within the problem. We will then create our constraints.

For each constraint, we will press “Add” to the right of the “Subject to the Constraints” box, which will be filled with our list of constraints. Within the “Add” window, we select the cell acting as the left hand side of our constraint as our “Cell Reference,” choose the inequality that is used within the constraint through the drop down, and then choose the cell acting as the right hand side of our equation as our “Constraint.” For this problem, we would also like for each variable amount to be an integer, as we cannot have a portion of a tire, so we will add integer constraints for each variable. To do this, we will add to our constraints. For each variable, we will choose their original reference cell (i.e., B1-B4) as their reference cell, and then choose “int” where we had previously put our inequality. Saving these, we now have all constraints added, including having each variable be an integer. Finally, we will use the drop down next to “Select a Solving Method” to choose “Simple LP,” as we are solving this problem through Simplex Linear Programming, and press solve. See Figure 3.

Figure 3: The set up in Excel's Solver

Solver Parameters

Set Objective:

To: ☒ Max ☐ Min ☐ Value Of:

By Changing Variable Cells:

Subject to the Constraints:

- \$B\$4 = integer
- \$B\$3 = integer
- \$B\$2 = integer
- \$B\$16 <= \$D\$16
- \$B\$15 <= \$D\$15
- \$B\$14 <= \$D\$14
- \$B\$13 >= \$D\$13
- \$B\$12 >= \$D\$12
- \$B\$1 = integer
- \$B\$10 >= \$D\$10
- \$B\$11 >= \$D\$11

☒ Make Unconstrained Variables Non-Negative

Select a Solving Method:

Solving Method

Select the GRG Nonlinear engine for Solver Problems that are smooth nonlinear. Select the LP Simplex engine for linear Solver Problems, and select the Evolutionary engine for Solver problems that are non-smooth.

Buttons: Add, Change, Delete, Reset All, Load/Save, Options, Help, Solve, Close

After pressing solve, Excel computed our optimal solution and placed the optimal values for each of our variables into their reference cells, B1:B4. We see that the optimal solution for this problem is to have 52 Tire A's, 9 Tire B's, 6 Tire B's, and 33 Tire D's at the store each week. Figure 4 shows how our final screen will look.

Figure 4: The solution in Excel

	A	B	C	D	E
1	x1	52			
2	x2	9			
3	x3	6			
4	x4	33			
5					
6					
7					
8	Objective	12750			
9					
10	Constraints	52 >=		5	Minimum tire amount of Tire A
11		9 >=		5	Minimum tire amount of Tire B
12		6 >=		5	Minimum tire amount of Tire C
13		33 >=		5	Minimum tire amount of Tire D
14		100 <=		100	Maximum total tire amount
15		70 <=		70	Maximum labor amount for Tires A & B
16		150 <=		150	Maximum labor amount for Tires C & D
17					
18					
19					
20					

From viewing our final solution in Excel, we can see how each constraint influenced the final solution to our problem. We can see that each of our three constraints that limited the total number of tires created, i.e., having the total number of tires at or below 100 and having the labor hours for the two groups of tires at or below certain values, were each maxed out and prevented any further tires from being produced. We also see that each of our constraints that prevented each tire production from being lower than 5 were not used as no tire amounts were at the minimum value of 5. Due to this being a problem of maximization, it makes sense that our limiting constraints were used, while our constraints that needed to be exceeded were exceeded naturally without the influence of the constraints.

Occasionally, a linear programming method will not have a possible solution. If this is the case when solving in Excel, the Solver will tell you after Solve is pressed, and will find the closest solution while breaking as few constraints as possible. Excel will also give you the option to view the answer and sensitivity reports, which gives further information about our final answers. This includes the solution to our objective equation, whether each constraint was binding, and the amount of slack left in each constraint. This information can be helpful



for further analysis. Following our example, the general idea of linear programming should now make sense and we can move on to the addition of chance constraints.

## Chapter 2: Introduction to Chance Constraints

Our above examples show cases of traditional linear programming problems; real world examples typically will not have all values within the problem's constraints or objective functions known. The most common example of this is having the right hand side values of the problem's constraint functions unknown. For instance, in the above example, it may instead be unknown what the true minimum value is for the amount of each tire needed to be stored, the most amount of tires that can be stored, or the total labor hours available. Rather, there may be a range of possible values that each constraint will fall within, or even an unknown range that has long tails to either side, before we eventually learn the true value later on after the optimization has been completed.

When the right hand sides of a constraint each contain a range of possible values instead of a single known value, one possible way to alleviate this is to use robust optimization. Using robust optimization, the most extreme values that are opposite of the operation will be used to ensure that the final solution will have the true values of the constraint satisfied and will exist within the feasible region. For example, if a constraint asks that the summation of values will be less than or equal to the right hand side, which is a range of possible values, using the minimum value within that range will ensure that if any other values within the range are the true value, the constraint will continue to be satisfied [6]. Though this method often results in conservative values being chosen for each variable or resource, this can often be necessary when working with constraints that cannot be broken without extreme consequences, meaning that it is in the best interest to solve using the most extreme constraint values in order to be certain that all possible values will be satisfied. Such constraints are

called “hard constraints.”

However, many of these fuzzy right hand side values do not have a clear range of values, and often can allow for small violations without severe consequences. These constraints with room for violation, or soft constraints, and unknown ranges for the right hand side can then be best solved through the use of chance constraints. Chance constraints determine a solution based on the likelihood that it will satisfy a certain probability of possible values for the constraints. For example, if the exact value of the right hand side value,  $b$ , is unknown, but is a random variable with a normal distribution of possible values, we are able to use  $\mu$  and  $\sigma$  of the random variable to find which value to use to ensure that the solution has a certain probability of satisfying the constraint, such as 95% or 99%. Thus, if the true value does happen to not fit within these large intervals, the soft constraint will allow for this small violation to be acceptable. Though these large probabilities will again often result in conservative constraint values similar to that of robust optimization, not taking into account the final values within the tails of the distribution will at least prevent overly conservative values based on very unlikely scenarios. Also note that in this paper, we will be focusing on the use of normal distributions for our random variables.

As we had previously seen, the general form of a linear programming constraint will align with the following:

$$\sum_{j=1}^n a_{i,j}x_j \leq b_i, \quad i = 1, \dots, m$$

The addition of the random variable for the right hand side,  $b$ , changes this to the following:

$$Pr(b_i \geq \sum_{j=1}^n a_{i,j}x_j) \geq \alpha, \quad i = 1, \dots, m.$$

With  $\alpha$  being the probability that is wished to be used, meaning that the probability

that the constraint is satisfied is greater than or equal to the value of  $\alpha$ .

As we are often working with normal distributions, satisfying this value of  $\alpha$  can be accomplished using the mean and standard deviation of the random variable  $b$ . By manipulating our above equation, we can reach the following:

$$Pr\left(\frac{b_i - \mu}{\sigma} \geq \frac{\sum_{j=1}^n a_{i,j}x_j - \mu}{\sigma}\right) \geq \alpha, \quad i = 1, \dots, m.$$

This now allows  $\frac{b_i - \mu}{\sigma}$  to be a random variable represented by the standard normal, meaning that it is now based on a normal distribution of  $\mu = 0$  and  $\sigma = 1$ . Furthermore, this constraint will hold if the following equation is satisfied.

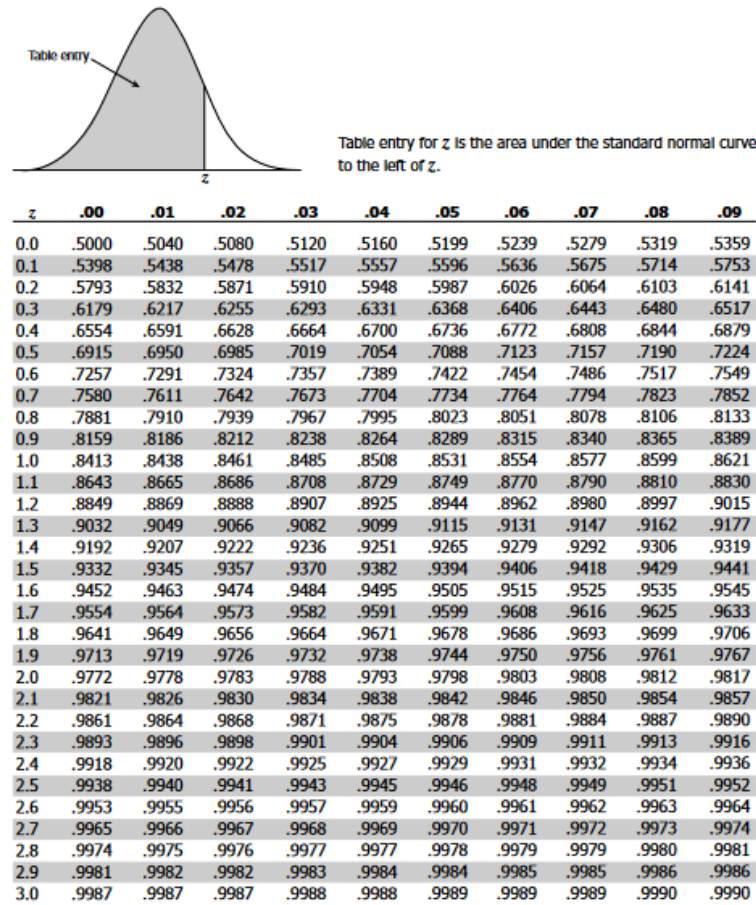
$$\left(\frac{\sum_{j=1}^n a_{i,j}x_j - \mu}{\sigma}\right) \leq Z_\alpha$$

Where  $Z_\alpha$  represents the number of standard deviations that the given value lies from  $\mu$ . This can be found from solving for  $Z_\alpha$  in the following integral:

$$\int_{Z_\alpha}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-x^2/2} dx = \alpha.$$

This integral equals the area,  $\alpha$ , of the normal distribution that lies between the value of  $Z_\alpha$  until the end (far right-tail) of the distribution, or infinity. Though this integral can be used to solve for  $Z_\alpha$  by plugging in the value of  $\alpha$  that is wished to be used for the probability, the complexity of the integral makes for difficult solving. As the values in the integral are not unique to a particular problem, the potential values for  $Z$  are often shown in a table to make for easy finding, such as the one below.

Figure 5: Table for a Normal Distribution [16]



To find the  $Z_\alpha$  associated with our chosen  $\alpha$  value from the table, we first find  $\alpha$  in the center of the table. Occasionally, our  $\alpha$  value will not be listed, and we will have to interpolate between the two closest values on either side of it. If  $\alpha$  is listed, we find the associated  $Z$  value by combining the  $Z$  value listed in the left-most column of our current row, and the above-most row of our current column. For example, if we are looking for the  $Z$  value associated with 97.5% probability, we find .9750 in the table and see that the  $Z$  value is 1.96.

Another way of finding the new right hand side values for our constraints is to use the function

`norm.inv()`

within Excel. In the cells in which we would have inputted our right hand side values for our equations, we can instead input the equation

`=norm.inv(rand(), mean, sd),`

with the probabilities, means, and standard deviations listed for that constraint, and it will give us the lower bound value for that probability.

When each individual constraint is given a certain probability that it must fall within, this is considered having single constraints, which is how we will solve the example below.

## Example Using Single Constraints:

Returning to our car shop, the shop contains the same constraints on tire amounts and labor hours, however these constraints now have unknown right hand sides due to variations in the amount of other supplies that may be in the store in a given week, and who will be working. Each right hand side value is based on a normal distribution, and we will look at each constraint one by one.

- When looking at the total number of tires that can fit in the shop, the mean number of tires that can fit is now 100, the original upper bound, with a normal distribution and a standard deviation of 4 about the mean. The new equation for this constraint will look as follows:

$$A + B + C + D \leq 100 + Z_{\alpha} \cdot 4$$

- The minimum number of tires that the store would like to hold each week will not be changed.
- The labor hours available for Tires A and B will now also be unknown, with a mean of 70 and a standard deviation of 2.5 (within a normal distribution). The new equation

for this constraint will look as follows:

$$A + 2B \leq 70 + Z_\alpha \cdot 2.5$$

- The labor hours available for Tires C and D are also now unknown, with a mean of 150 and a standard deviation of 6.7 (within a normal distribution). The new equation for this constraint will look as follows:

$$3C + 4D \leq 150 + Z_\alpha \cdot 6.7$$

Note that, though we are adding our  $Z_\alpha \cdot \sigma^2$  value to  $\mu$ , our  $Z_\alpha$  will be the negated value of what is found in the table, as we are looking at the proportion of values greater than our  $\alpha$  probability. Therefore, this term will actually be subtracted from our  $\mu$  rather than added.

We are again looking to find the number of each tire type that should be stored at the tire shop each week that satisfies all constraints, while maximizing profit. However, as each constraint has a long tail of possible values for the right hand side, we will not be able to be certain that the constraints will be satisfied 100% of the time. For this example, we will look at each constraint having a 95% probability that they will be satisfied ( $\alpha = 0.95$ ). Though this is not a plausible method for hard constraints, 95% probability is typically a satisfactory value for many soft constraints, and this value can be adjusted for different problems depending on the severity of consequences for breaking constraints. Note, however, that an increase in the probability that a constraint is satisfied will increase the range that we are allowing for our value to fall within, making our solution more conservative and less specific.

To find the  $Z$  value associated with the  $\alpha = 0.95$ , we return to the table in Figure 7. We see that this value is not listed, so we instead must interpolate the values associated with

alphas 0.9495 and 0.9505. We see that the Z value associated with an alpha of 0.9495 is 1.64 and with alpha 0.9505 is 1.65, so we are able to interpolate these two points to conclude that the Z value associated with the alpha of 0.95 is 1.645. This means that there is a 95% probability that the true value will be less than 1.645 standard deviations from our mean. We will now add this value to our constraints and construct a list of our finalized constraints.

- $A + B + C + D \leq 100 - 1.645 \cdot 4$
- $A + 2B \leq 70 - 1.645 \cdot 2.5$
- $3C + 4D \leq 150 - 1.645 \cdot 6.7$

Final Constraints:

- $A + B + C + D \leq 93.42$
- $A \leq 5$
- $B \leq 5$
- $C \leq 5$
- $D \leq 5$
- $A + 2B \leq 65.8875$
- $3C + 4D \leq 138.9785$

Now, our constraints look like typical linear programming constraints, and we can input our problem into Excel as we had done previously.



Figure 6: Setup in Excel

	A	B	C	D	E	F
1	x1					
2	x2					
3	x3					
4	x4					
5						
6						
7						
8	Objective	=60*B1 + 90*B2 + 150*B3 + 240*B4				
9						
10	Constraints	=B1	>=	5	Minimum tire amount of Tire A	
11		=B2	>=	5	Minimum tire amount of Tire B	
12		=B3	>=	5	Minimum tire amount of Tire C	
13		=B4	>=	5	Minimum tire amount of Tire D	
14		=B1 + B2 + B3 + B4	<=	93.42	Maximum total tire amount	
15		=B1+2*B2	<=	65.8875	Maximum labor amount for Tires A & B	
16		=3*B3+4*B4	<=	138.9785	Maximum labor amount for Tires C & D	
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						

Solving, we will get our revised final solution.

Figure 7: Solution in Excel

	A	B	C	D	E	F	G	H
1	x1	49						
2	x2	8						
3	x3	6						
4	x4	30						
5								
6								
7								
8	Objective	11760						
9								
10	Constraints	49	>=	5	Minimum tire amount of Tire A			
11		8	>=	5	Minimum tire amount of Tire B			
12		6	>=	5	Minimum tire amount of Tire C			
13		30	>=	5	Minimum tire amount of Tire D			
14		93	<=	93.420585	Maximum total tire amount			
15		65	<=	65.8875	Maximum labor amount for Tires A & B			
16		138	<=	138.9785	Maximum labor amount for Tires C & D			
17								
18								
19								
20								

Our adjusted optimal solution is now to have 49 Tire A's, 8 Tire B's, 6 Tire C's, and 30 Tire D's, for a weekly profit of \$11,760. Note that, again, each of our limiting constraints

were maximized, while none of our constraints requiring a maximum were needed. The only tire that did not see a decrease in total was Tire C, while the profit also decreased by almost \$1,000. This method of solving within Excel was also described in Hillier and Lieberman [6].

Though we decreased our profit when we lowered our right hand side values, we raised the probability that each chance constraint will be satisfied from 50% (when only values less than the mean would be satisfied), to 95%. However, this confidence decreases when each constraint contains this probability. When two constraints each have a 95% probability of being satisfied, the probability that both will be satisfied is only  $.95 \cdot .95 = 0.9025$ , or 90.25% probability. For this example, since three constraints each have a 95% probability of being satisfied, the probability of all constraints being satisfied becomes  $.95^3 = 0.8574$ , or 85.74% probability. Each increase in the number of chance constraints used only further lowers this probability of all constraints being satisfied. It is often advised to use a much higher probability value when working with many constraints to keep the total probability as high as possible, though this will lower the value of the optimal solution even further.

## **Joint Constraints:**

As mentioned above, taking individual probabilities for each constraint with a random variable to the right hand side will greatly decrease the overall probability that each constraint is satisfied with the addition of each such constraint. We saw that for our above problem, the probability that all three such constraints will be satisfied using the above method of single constraints drops from 95% to 85.74% when satisfied together, and would only continue to drop as more are added. As mentioned, one solution can be to simply increase the individual probabilities to a value that is high enough to allow for our overall probability to be within an acceptable range. For our above problem, if we instead preferred for the overall probability to be 95%, we can take the cube root of .95 to see what individual probabilities will be needed to make this possible.

$$\sqrt[3]{.95} = .97468$$

Thus, if each single constraint was solved using 97.468% probability, the overall probability that all three constraints will be solved will be 95%.

However, an alternative method for looking at the constraints as a combined probability is the method of joint constraints. This method is typically necessary when the constraints are not independent, so the above method of simply adjusting  $\alpha$  to satisfy an overall probability will not account for the relationships between constraints values. Attempting to account for this relationship adds great complexity to the problem, which we will illustrate in the example below.

**Example:** A seamstress is hand-knitting two versions of the same shirt that she would like to sell, each made from yellow and orange yarn. The long sleeve shirt that she is making will sell for \$48, but requires additional materials compared to the short-sleeve shirt she is designing, which will be sold for \$39. Each long-sleeve shirt will require 2 yards of yellow yarn and 1.5 yards of orange yarn, whereas each short-sleeve shirt will require 1.45 yards of yellow yarn and 1.15 yards of orange yarn. The only constraints that she has on the amount of shirts she can make each week is the amount of yarn that she has available, which will often be unknown until the start of the week. If the availability of each yarn type is a random variable with normal distribution of  $\mu = 15$  yards and  $\sigma = 1.1$  for yellow yarn and  $\mu = 12$  yards and  $\sigma = .9$  for orange yarn, what is the optimal number of each shirt that she should knit each week to maximize profit and allow her to be 95% confident that each week she will have enough yarn for the number of shirts that she plans to make?

**Discussion of Solution:** At first glance, this example seems to be able to be solved similarly to our most recent example, with the exception of our  $\alpha$  being the square root of .95 to allow for the overall probability to be .95 for both shirts. However, as both of our constraints are related to the availability of similar resources, it is not unreasonable to

assume that the weekly availability of these resources are related to one another and their probabilities are not independent. In fact, this is often true even for resources that may not be as similar to one another as those in this example, as supply chain issues that affect supply amount are often far-reaching.

Before solving, we will first visualize the set up of the problem without the addition of joint constraints.

Variables: Number of each shirt type to make

- Long-sleeve shirt:  $x_1$
- Short-sleeve shirt:  $x_2$

Objective (Maximize Profit):  $Z = 48x_1 + 39x_2$

Constraints:

- At most 15 yards of yellow yarn:  $2x_1 + 1.5x_2 \leq b_1$
- At most 12 yards of orange yarn:  $1.45x_1 + 1.15x_2 \leq b_2$

When previously solving using typical chance constraints, we would have begun solving this problem by finding the  $Z$  values needed for each right-hand side of our two constraints, using the below probability.

$$P\left(\sum_{j=1}^n a_{ij}x_j \leq b_i\right) \geq \alpha, \quad i = 1, 2$$

However, due to non-independence, the two separate probabilities that we had for our two constraints must now be combined into a single probability.

$$P\left(\sum_{j=1}^n a_{1j}x_j \leq b_1 \text{ and } \sum_{j=1}^n a_{2j}x_j \leq b_2\right) \geq \alpha$$

Our previous probability would have been able to be solved for by converting the normal distribution into the standard normal and simply finding the  $Z_\alpha$  value from the table as

mentioned before, which is no longer possible by the existence of two constraints at once. The solving of this joint probability leads to a non-linear vector constraint based on a product of multi-variate normal distributions.

Reworking the earlier-mentioned equation that found the value of  $Z_\alpha$  from the integral for the area under the normal distribution, the multi-variate normal pdf is:

$$f(\vec{x}) = \frac{1}{\sqrt{2\pi} |\Sigma|} e^{-(\vec{x}-\vec{\mu})^T \Sigma (\vec{x}-\vec{\mu})}$$

With the additional components being the following:

- $\vec{x}$  is the vector of random variables
- $\vec{\mu}$  is the vector of means of the random variables
- $\Sigma$  is the variance-covariance matrix with the diagonal values being  $\sigma_i$ , the standard deviation for the random variables, and the off-diagonal elements being  $\sigma_{ij}$ , representing the covariance of  $x_i$  and  $x_j$
- $|\Sigma|$  represents the determinant of  $\Sigma$

Solving this is extremely difficult, and requires additional knowledge on the relationship between  $x_i$  and  $x_j$ . This complexity has resulted in this method being rarely used unless necessary, such as in financial settings when non-independence between constraints cannot be ignored. However, despite the complexity and rarity of its use, the allowance that joint constraints have given to the solving of these linear programming problems with non-independent chance constraints has cemented the importance of joint constraints within our overall discussion of chance constraints.

## Chapter 3: Historical Development of Chance Constraints

As previously mentioned, the development of operations research and linear programming first became formalized during the 1940's, as wartime operations required a more definite method for determining optimum efficiencies in the creation of wartime supplies. As these methods quickly spread to other industries, it was then realized soon after that methods for handling constraints with uncertain values within linear programming problems needed to be developed, as well.

Several methods have since been developed to handle different variations of constraints, with robust optimization being a notable further step. Though not developed until relatively recently, robust optimization alleviates issues related to problems containing hard constraints whose values exist within set ranges. However, robust optimization was not able to combat issues related to soft constraints with large distributions, in which long tails on either side of the distribution prevented a definite range of values from being available to minimize. Though soft constraints do allow for violations, it was still necessary to create a method that kept these constraint values that may exist within these indefinite values as close to satisfied as possible. This resulted in the development of chance constraints [6].

The concept of chance constraints was first introduced by mathematicians A. Charnes and W. W. Cooper in a 1959 paper titled "Chance-Constrained Programming" [2]. In their six page paper, Charnes and Cooper describe "chance-constrained programming," or linear programming using chance constraints, as a method to solve problems containing "random variables with known distributions in such a manner as (a) to maximize a functional of

both classes of random variables subject to (b) constraints on these variables which must be maintained at prescribed levels of probability.” Charnes and Cooper use a two-step method and an example involving tankage facilities and their supplies of oil to showcase the workings of this method. Though there has been mention of other mathematicians possibly having invented a similar method previously to this, this is the most clear definition given up until this point, and led the way to the method becoming commonly-used to solve problems of this form [5].

Though “Chance-Constrained Programming” was the first official mention and coining of chance constraints, the paper makes reference to four earlier works that laid the groundwork to this development. The most notable of these is the 1958 paper “Cost Horizons and Certainty Equivalents: An Approach to Stochastic Programming of Heating Oil,” written also by Charnes and Cooper, as well as G. H. Symonds [1]. This paper describes a similar approach to that of “Chance-Constrained Programming,” in which probability is used to determine an acceptable value for each constraint. A note in this paper about being the first in a series of papers to discuss this topic also gives insight into the ultimate goal of the publication- to set the scene for the eventual release of “Chance-Constrained Programming” in the following year. Comparing the lengths of these two papers as well, with the 1958 release being 27 pages and “Chance-Constrained Programming” being only 6, may also demonstrate that the initial release was intended to provide initial descriptions to chance constraints in order to allow the official introductory paper to be as concise as possible. Their 1958 paper also makes reference to 22 former papers, 8 of which co-written by A. Charnes, further showcasing the amount of previous work that led to the eventual coining in 1959.

Charnes and Cooper released several other papers further discussing chance constraints in the years following the release of their 1959 introductory paper, including the 1962 paper “Chance Constraints and Normal Deviates” [3] and the 1963 paper titled “Deterministic Equivalents for Different Objectives in Chance-Constrained Programming” [4]. These each

discuss different applications of chance-constrained programming, as well as delve further into specific topics related to chance constraints. This continued release of papers by Charnes and Cooper related to chance constraints illustrates the value that the two mathematicians believed that the method had within mathematics, and the work that they were willing to put forth in order to cement it within the field.

During this time, other mathematicians within the field of operations research began releasing their own papers utilizing and making reference to these concepts, as well. The 1961 paper “Some Reflections on Static Programming under Uncertainty” by H. Theil [15] discusses the complicated and complex nature of linear programming with uncertain constraints, based on the idea proposed thus far of chance constraints and including examples of several variations of possible problems that could exist. However, though he acknowledges the complexity of the method, he also begins by making reference to the heavy focus that there was at the 1957 Stockholm Conference of the International Statistical Institute on wanting a method to solve programming problems with uncertainty. This reference showcased the overall frame of mind at the time in relation to this method; there was a clear need for such a method within linear programming, and though it came with complexities, Charnes and Cooper’s development in 1959 created a solution.

A look at the number of papers that include references to chance constraints further illustrates the speed in which the method spread through different fields- with papers ranging in content from further improvements or opinions on the method, to different applications. The 1963 paper “Minimum-Cost Cattle Feed under Probabilistic Protein Constraints” by C. van de Panne and W. Popp [12] is an example of this, in which chance constraints are used within the traditional linear programming problem of cattle feed. The example also includes additional variations to the initial discussion of chance constraints, such as having the problem be static rather than dynamic. As one of the first papers to be released with the overall focus being on the utilization of chance constraints, this application showcased to others the practicality of the method, and made way for the normalization of chance



constraints in solving linear programming problems with uncertain constraints. In the year of its release, nearly 200 papers related to chance constraints were released, while 1970 saw a jump to almost 700. This notable continues to increase over time, with 1980 seeing over 2,100 papers and 1990 seeing over 7,700 with mentions to chance constraints. Currently, almost 17,000 papers have been released with a mention of chance constraints in this year alone.

Though several of the above papers added additional insight and leverage to the development of chance constraints, the next paper considered to impact the overall development of the method of chance constraints following the initial coining was the 1965 paper “Chance-Constrained Programming with Joint Constraints” by B. L. Miller and H. M. Wagner [10]. As mentioned above, the addition of joint constraints allowed for complex linear programming problems with non-independent chance constraints to be solved for by using a singular probability that needed to be satisfied by all constraints as a whole. Though complicated, this allowed for a way for these more complex problems to be solved, which increased the applicability and overall usage of chance constraints. Prékopa is also often credited with helping develop joint constraints, and later releases the paper “On probabilistic constrained programming” [14] in 1970, which proves topics related to the convexity of the feasible region. Issues related to the complexity of convexity within chance constraint problems is considered one of the major drawbacks to the method, and having any additional information on how the convexity of the problems worked was greatly beneficial. This description of the origins of chance constraints is also further described in the dissertation “Chance constrained problem and its applications” [13].

In the years following its rapid initial developments, few major developments have since been made to the method, with instead focus being on the improvement and applications within individual fields. Just the development of joint constraints for the handling of non-independent variables provided a new range of complex uses of the method, allowing chance constraints to be employed in a much broader range than traditionally considered. Currently,

new developments are focused on utilizing chance constraints in other processes, such as process-engineering and chemical-engineering problems under uncertainty [8]. Each of these further discoveries illustrates the importance and range that chance constraints can have throughout numerous fields, and how necessary the initial discovery by Charnes and Cooper continues to be today.

## Chapter 4: Applications of Chance Constraints

As we have seen mentioned above, there has been plenty of discussion on the numerous uses and potential uses that chance constraints can have since Charnes and Cooper’s first discovery of the method in 1959. From calculating the minimum cattle-feed cost in 1963 to current explorations within chemical engineering feats, there is plenty to discuss in terms of the overall applications of chance constraints within an array of real-world sectors.

The most classical applications of chance constraints include their use in energy allocation, water reservoir management, and risk management, with energy allocation often being considered the most common application. Almost all forms of energy allocation, whether from renewable or nonrenewable resources, require chance constraints to ensure that the necessary amounts of resources will be available for consumers at all times. Though sustainable resources are renewable by nature, many random variables exist in terms of the amount of resources available at any given time. For example, when determining the optimal amount of energy that can be obtained from a wind farm or a dam, variations in wind speed or rain water amount cannot be determined with certainty, and probabilities must instead be used to give a certain level of confidence in their productions of energy, without over-promising. These probabilities in amount of resources are then computed with energy demand to determine with a certain amount of confidence the optimal amount of each resource that should be expected [8]. APPsearch alone shows over 700,000 articles that include references to both chance constraints and energy, illustrating how widely-used this method is for determining these solutions.

Similarly, water reservoir management and risk management both contain numerous

potentially-unknown variables, with great emphasis by many on determining their optimal solutions. In terms of water reservoirs, there are great consequences for not determining the correct threshold that water can reach within a reservoir, while great variability exists in the possible amount of water that could be present in one at any given time. The opposite can also be true, with instead of the risk of overflowing being the main concern, instead the risk of water scarcity and not having enough water available for the overall demand is often just as concerning [8]. An excellent example of the latter is the 2017 paper “Chance-Constrained Dynamic Programming for Multiple Water Resources Allocation Management Associated with Risk-Aversion Analysis: A Case Study of Beijing, China” [9], which uses chance constraints to look at the high risk of potential water scarcity within Beijing, China to determine the best solution to water resource allocation. Overall within water reservoir management, chance constraints are used to determine the correct size that a water reservoir should be, while minimizing the optimal solution of cost and using probability to ensure to a certain confidence that the threshold for water will not be exceeded, while not being certain of the potential water amounts that could be present.

In terms of the use of chance constraints within risk management, there is of course much volatility within the stock market and when predicting overall market conditions, making it impossible to be certain that an investor will not end up losing money within any given investment [8]. However, past knowledge can be used to determine within a certain probability the likelihood that certain events will take place, allowing an optimal decision to be made in terms of the amount of money that should be invested at any given time into certain stocks. Though investment companies will formally use the method of chance constraints in these situations, almost everyone that invests uses a loose version of chance constraints to try to determine the amount of risk that is associated with any potential investment, and using the to determine the optimal amount that should be invested. From there, this loose version of chance constraints can be applied to almost all instances of people weighing potential risks to determine their optimal solution to a situation.

In addition to the traditional applications of chance constraints, there have been many uses of the method within technology that have begun appearing in the past several years, which illustrate the direction that chance constraint applications are moving towards in the future. As we have recently seen an exponential rise in the different uses and possibilities of technology, it is no surprise that the use of chance constraints is not far behind. With each additional push in technology, often additional room for error is introduced that must be accounted for. For example, the innovation of self-driving cars included finding solutions to the many random variables that can exist when driving, such as any potential obstacle that can occur in the road or errors in any of the cars mechanics. Chance constraints are used to determine the likelihood of these different events occurring, and factoring this into determining how these obstacles and errors will be avoided [7]. Even space exploration utilizes chance constraints to handle a number of unknown variables, from problems regarding the take-off and landing of space crafts to determining their optimal flight path [11]. Each of these problems quickly become exponentially more complex than the traditional examples using resource allocations that we originally looked at, but they illustrate the direction that chance constraints are moving in, and the potentials that they have.

Overall, chance constraints are an integral part of many fields, used to understand and account for risk in countless scenarios. The potential uses of the method has expanded greatly in the over 63 years since its initial inception, and it is clear that there are many more potential uses yet to be discovered.

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## **Vita**

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